Team Contest

Junior League

1. A trapezoid $ABCD$ with bases AB and CD is such that the circumcircle of the triangle BCD intersects the line AD in a point E, distinct from A and D. Prove that the circumcircle of the triangle ABE is tangent to the line BC.

2. Is it possible to choose three numbers from $\frac{1}{100}, \frac{2}{99}, \frac{3}{98}, \ldots, \frac{100}{1}$ so that their product is equal to 1?

3. Numbers $1, 2, \ldots, n$ are written in some order in a single line. A pair of numbers is called a *hole* either if this numbers are next to each other or if all the numbers in between them are less than each of them. What is the maximum possible number of holes? (One number may be included in several holes.)

4. Several chords are drawn in a circle so that every pair of them intersects inside the circle. Prove that all the draw cords can be intersected by the same diameter.

5. Natural numbers a and b satisfy $2a^2 + a = 3b^2 - b$. Prove that $a + b$ is an exact square.

(None)

6. The point I_b is the center of an excircle of the triangle ABC, that is tangent to the side AC. Another excircle is tangent to the side AB in the point C_1. Prove that the points B, C, C_1 and the midpoint of the segment BI_b lie on the same circle.

7. Written on a blackboard is the polynomial $x^2 + x + 2014$. Sasha and Fedya take turns alternately (starting with Fedya) in the following game. At his turn, Fedya must either increase or decrease the coefficient of x by 1. And at his turn, Sasha must either increase or decrease the constant coefficient by 1. Fedya wins if at any point in time the polynomial on the blackboard at that instant has integer roots. Prove that Fedya has a winning stratergy.

8. On a circle 35 numbers are written. Every two adjacent numbers differ by no more than 1. Prove that the sum of squares of all the numbers is at least 10.

9. For which n can a convex n-gon be split into convex hexagons?

Senior League

1. Is it possible to choose five numbers from $\frac{1}{100}, \frac{2}{99}, \frac{3}{98}, \ldots, \frac{100}{1}$ so that their product is equal to 1?

2. Numbers $1, 2, \ldots, n$ are written in some order in a single line. A pair of numbers is called a *hole* either if this numbers are next to each other or if all the numbers in between them are less than each of them. What is the maximum possible number of holes? (One number may be included in several holes.)

3. The point I_b is the center of an excircle of the triangle ABC, that is tangent to the side AC. Another excircle is tangent to the side AB in the point C_1. Prove that the points B, C, C_1 and the midpoint of the segment BI_b lie on the same circle.

4. Prove that for any sequence of positive numbers $a_1, a_2, \ldots$ there exists a number n such that $\frac{1+a_{n+1}}{a_n} \geqslant 1 + \frac{1}{n}$.

5. The spheres S_1, S_2 and S_3 are externally tangent to each other and all are tangent to some plane at the points A, B and C. The sphere S is externally tangent to the spheres S_1, S_2 and S_3 and is tangent to the same plane at the point D. Prove that the projections of the point D onto the sides of the triangle ABC are vertices of an equilateral triangle.

6. Through the center of an equilateral triangle ABC an arbitrary line l is drawn that intersects the sides AB and BC in points D and E. A point F is constructed, so that $AE = FE$ and $CD = FD$. Prove that the distance from F to the line l does not depend on the choice of l.

7. For which n can a convex n-gon be split into convex hexagons?

8. The polynomial $f(x)$ has real coeffcients and is of degree $2n-1$, and $(f(x))^2 - f(x)$ is divisible by $(x^2 - x)^n$. Find all possible values of the leading coefficient of f.

9. There are $n > 1$ balls in a box. Two players A and B are playing a game. At first, A can take out $1 \leqslant k < n$ ball(s). When one player takes out m ball(s), then the next player can take out l balls, where $1 \leqslant l \leqslant 2m$. The person who takes out the last ball wins. Find all positive integers n such that B has a winning strategy.